Nucleon Form Factors from 5D Skyrmions

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Abstract

Several aspects of hadron physics are well described by a simple 5D effective field theory. Baryons arise in this scenario as "large" (and therefore calculable) 5D skyrmions. We extend and refine the existing analysis of this 5D soliton, which is fairly non-trivial due to the need of numerical methods. We perform the complete quantization of those collective coordinates which are relevant for computing the static observables like the nucleon form factors. We compare the result with simple expectations about large-N_c QCD and with the experimental data. An agreement within 30% is found.

1 Introduction and Conclusions

Certain 5D effective gauge theories, often referred to as "Holographic QCD" or "AdS/QCD" models [1–4], closely resemble low-energy QCD in the limit of large number of colors N_c. The similarity is qualitative as these theories contain, like large-N_c QCD, infinite towers of weakly interacting mesons, but also quantitative. Leading order calculations in such 5D models typically describe the physics of the lightest mesons to 10% accuracy in terms of an extremely limited number of parameters. These results are compatible with the hypothesis that leading order calculations in the 5D model reproduce large-N_c QCD.

Baryons arise in this scenario as solitons with a conserved topological charge which represents the baryon number. They are the 5D analog of skyrmions [5,6] (see [7] for a review), so we will refer to them as 5D skyrmions. They differ from ordinary skyrmions, however, in the important aspect of calculability, as shown in [8,9]. The skyrmion solutions obtained in 4D models of mesons –which may or may not contain some vector resonances on top of the pion field– have a size which is of the order of the inverse cut-off of the theory, and incalculable UV effects do not decouple. This problem, which constitutes the main reason of theoretical dissatisfaction about the 4D Skyrme model, is solved in the 5D case because the size of the 5D skyrmion is larger than the 5D cut-off.

A string construction named "Sakai–Sugimoto model" [10] might provide a UV completion of the AdS/QCD models and give them an interpretation in the standard framework of AdS/CFT. In the low-energy supergravity limit of large 't Hooft coupling $\lambda \to \infty$ the Sakai–Sugimoto model reduces indeed to a 5D theory with $U(N_f)$ gauge symmetry (N_f denotes the number of flavors) and two AdS$_5$-like boundaries on which the sources for Left- and Right-handed currents are located. This can be rewritten as a $U(N_f)_L \times U(N_f)_R$ theory living on one-half of the space with one AdS$_5$-like (UV) boundary on which both sources live and one IR boundary on which symmetry-breaking conditions as in eq. (2) are imposed. The Sakai–Sugimoto model is almost equivalent, in the limit in which practical calculations are performed, to the effective theory considered in the present paper. There is however a difference which, as remarked in [8, 9], becomes extremely relevant in the baryon sector. In the Sakai–Sugimoto model the effective 5D interaction scale M_5 is proportional to the 't Hooft coupling λ, *i.e.* $M_5 \propto \lambda$, while the coefficient of the Chern–Simons (CS) term, which is fixed by the Adler–Bardeen anomaly, has no λ factor. This implies, given the definition in eq. (7), $\gamma \propto 1/\lambda \to 0$. The parameter γ controls the size of the skyrmion, $\rho \propto \gamma^{1/2}$ in the Sakai–sugimoto model and for this reason the string effects, which are encoded in higher dimensional operators, do not decouple and there is no advantage with respect to the usual

4D Skyrme model in what concerns calculability. It is interesting, nevertheless, to forget about higher dimensional operators and study baryons in this framework, describing them in terms of "small" Yang–Mills instantons [11] or in terms of 4D Skyrmions [12].

In the present paper we complete and refine the analysis of the 5D skyrmions presented in [9], with the aim of computing nucleon static observables and in particular the current form factors. To this end we need to perform a complete quantization of the 5D skyrmion collective coordinates, which is a non-trivial task as it requires to solve numerically a new set of partial differential equations. Obtaining predictions for the form factors at non-zero transfer momentum requires, moreover, an increased precision of the solution, which we obtain by refining our numerical method. The nucleon form factors in the Sakai–Sugimoto model have been computed in [13] (see also [14,15]), by performing a "small-size" (*i.e.* small γ) expansion in which analytical results can be obtained. [1] This expansion is not trustable in our case because, as explained above, the size of the 5D skyrmions is large and $\gamma \sim 1$.

The paper is organized as follows. In sect. 2, after a brief review of the model and of the static 5D skyrmion solution we identify the zero-mode fluctuations which are relevant to describe static properties and we discuss the corresponding collective coordinates classical Lagrangian. A suitable ansatz is described which permits to rewrite in a 2D form the 4D equations which define the zero-modes. Sect. 3 is devoted to the collective coordinate quantization and to the calculation of the form factors, this discussion is basically the same as in the 4D Skyrme model [6,7], though adapted to the present case. Sect. 4 contains a detailed presentation of our results. After the comparison with simple expectations about large-N_c QCD we discuss the divergences due to the chiral limit and we check that the Goldberger–Treiman relation holds in our model. Finally, we compare our results with experimental data and find a level of agreement better than 30% for all the observables, with the notable exception of the axial coupling g_A for which we find $g_A = 0.70$ versus an experimental value $g_A = 1.25$.[2] Most of the technical details are presented in the appendices. In appendix A the 2D equations of motions and boundary conditions are derived while appendix B gives some detail on the numerical techniques we employed to obtain the solution.

In spite of the failure in the axial coupling, the level of accuracy of our results is consistent with the expected size of the $1/N_c$ corrections or, which is the same, the expected size of next-to-leading contributions in our model. It is not unreasonable that anomalously large

[1] The same calculation has been performed in [16] with a different (and erroneous, in our understanding) definition of the chiral currents.

[2] An erroneous value of g_A was reported in [9]. The error was due to a subtlety, which we will discuss in the following, in taking the zero momentum limit of the axial form factor, combined with a more trivial mistake.

numerical factors could change into 80% the naively expected 30% correction to g_A. Such large $1/N_c$ corrections arise for example if one follows, instead of the approach we consider, the quantization procedure of the collective coordinates proposed in [17]. This "alternative" quantization is equivalent to the standard one at the leading order in $1/N_c$ but it also contains large $1/N_c$ corrections. We will discuss in sect. 4 how these corrections change our predictions. In the case of g_A we find, remarkably, the much better result $g_A = 1.17$ while the level of the agreement of the other observables is unaffected. [3] Even without applying this correction, the results which we obtain are significantly more accurate than those of the original Skyrme model [6] (in which, we remark, g_A is also small, $g_A = 0.65$), but not as good as those of more refined skyrmion models (which, of course, also have more parameters) like the ones reviewed in [7]. It seems, as we will discuss in sect. 4, that the inclusion of the explicit breaking of the chiral symmetry (*i.e.* of the pion mass m_π) will improve the agreement of several observables and that g_A may display an enhanced sensitivity to m_π. It is certainly worth exploring this direction.

2 Skyrmions in 5D

2.1 The Model

We will consider the same model as in [9], *i.e.* a $U(2)_L \times U(2)_R$ gauge theory in five dimensions with metric $ds^2 = a(z)^2 \left(\eta_{\mu\nu} dx^\mu dx^\nu - dz^2\right)$, where we denoted as x^μ the usual 4 coordinates with mostly minus metric and with z, which runs in the interval $[z_{\rm UV}, z_{\rm IR}]$, the extra dimension. We choose our metric to be AdS$_5$ and therefore the warp factor $a(z)$ to be

$$a(z) = \frac{z_{\rm IR}}{z}\,, \tag{1}$$

with $z_{\rm UV} \to 0$ to be taken at the end of the calculations. In this limit, $z_{\rm IR}$ coincides with the conformal length $L = \int_{z_{\rm UV}}^{z_{\rm IR}} dz = z_{\rm IR} - z_{\rm UV}$. It should be kept in mind that, since gravity is non-dynamical in our model, the choice of the warp factor $a(z)$ is arbitrary. It is commonly believed, however, that different "reasonable" choices of $a(z)$ would not affect in a significant way the predictions for IR observables, like those we will compute in this paper.[4] Choosing AdS$_5$ –or at least a geometry with an "AdS$_5$-like" boundary– is crucial, on the contrary, if

[3] We thank the referee of Nucl. Phys. A for suggesting this possibility to us.

[4] This belief is supported by Ref. [18], in which IR predictions for flat and AdS$_5$ spaces were compared, and by Ref. [3], in which departures from AdS$_5$ were considered. Moreover, the Sakai–Sugimoto model is equivalent, for what calculations in the meson sector are concerned, to a 5D model of the kind we are considering with non-AdS$_5$ warped metric. The predictions of this model are very similar to those of AdS/QCD, again suggesting that the choice of the metric is not so relevant.

one wants to match UV correlation functions with those computed in QCD by perturbation theory [1–4, 18]. For this reason, in the literature the choice in eq. (1) is commonly adopted.

We will denote the $U(2)_L$ and $U(2)_R$ gauge connections respectively by $\mathbf{L}_M$ and $\mathbf{R}_M$, where $M = \{\mu, 5\}$, and parametrize them as $\mathbf{L}_M = L^a_M \sigma_a/2 + \widehat{L}_M 1\!\!1/2$ and $\mathbf{R}_M = R^a_M \sigma_a/2 + \widehat{R}_M 1\!\!1/2$ in terms of the Pauli matrices σ_a and the identity $1\!\!1$. Chiral symmetry is broken at the $z = z_{\rm IR}$ boundary (IR-boundary) by the following conditions:

$$(\mathbf{L}_\mu - \mathbf{R}_\mu)\,|_{z=z_{\rm IR}} = 0\,, \qquad (\mathbf{L}_{\mu 5} + \mathbf{R}_{\mu 5})\,|_{z=z_{\rm IR}} = 0\,, \tag{2}$$

where the 5D field strength is defined as $\mathbf{L}_{MN} = \partial_M \mathbf{L}_N - \partial_N \mathbf{L}_M - i[\mathbf{L}_M,\, \mathbf{L}_N]$, and analogously for $\mathbf{R}_{MN}$. On the other boundary, the UV one, we impose Dirichlet conditions:

$$\mathbf{L}_\mu\,|_{z=z_{\rm UV}} = 0\,, \qquad \mathbf{R}_\mu\,|_{z=z_{\rm UV}} = 0\,. \tag{3}$$

The 5D action $S = S_g + S_{CS}$ consists of a standard gauge kinetic part

$$S_g = -\int d^4x \int_{z_{\rm UV}}^{z_{\rm IR}} dz\, a(z)\, \frac{M_5}{2} \left\{ {\rm Tr}\left[L_{MN} L^{MN}\right] + \frac{\alpha^2}{2} \widehat{L}_{MN} \widehat{L}^{MN} + \{L \leftrightarrow R\} \right\}, \tag{4}$$

and of a Chern–Simons part

$$S_{CS} = \frac{N_c}{16\pi^2} \int d^5x \left\{ \frac{1}{4} \epsilon^{MNOPQ} \widehat{L}_M\, {\rm Tr}\left[L_{NO} L_{PQ}\right] + \frac{1}{24} \epsilon^{MNOPQ} \widehat{L}_M \widehat{L}_{NO} \widehat{L}_{PQ} - \{L \leftrightarrow R\} \right\}. \tag{5}$$

The S_{CS} is needed to reproduce the QCD anomalies and its coefficient is fixed to be proportional to the number of colors N_c.

In order to compare our 5D model with the real world, and in particular to compute the form factors as we will do in this paper, we need to identify the chiral currents to which the electroweak bosons are coupled. These operators, which would be given in QCD by the quark bilinears $j^a_{\mu, L\,(R)} = \overline{Q}_{L\,(R)} \gamma^\mu \sigma^a/2 Q_{L\,(R)}$, $\widehat{j}_{\mu, L\,(R)} = \overline{Q}_{L\,(R)} \gamma^\mu 1\!\!1/2 Q_{L\,(R)}$ correspond in our model to [9]

$$J^a_{L\,\mu} = M_5\big(a(z) L^a_{\mu\,5}\big)\,|_{z=z_{\rm UV}}\,, \qquad \widehat{J}_{L\,\mu} = \alpha^2 M_5 \big(a(z) \widehat{L}_{\mu\,5}\big)\,|_{z=z_{\rm UV}}\,, \tag{6}$$

and analogously for R.

It is important to remark that this model, as discussed in [9], is a valid effective field theory with an NDA cut-off Λ_5 which is bigger than the scale of the lightest resonances and can be sent to infinity for $M_5 \to \infty$. This allows us to include only the lowest dimensional operators in the action (4,5) since the others, which would surely arise in a UV completion of

the model, are expected to give a subleading contribution. At the leading order this model is extremely predictive: its only 3 parameters are M_5, L and α. The 5D interaction scale M_5 can be traded for the adimensional parameter

$$\gamma = \frac{N_c}{16\pi^2 M_5 L\alpha}\,, \tag{7}$$

which controls the size of the skyrmion; $\rho \sim \gamma^{2/3}$. We want to interpret the 5D weak coupling expansion as the $1/N_c$ expansion, therefore we will take the interaction scale to scale like N_c, *i.e.* $M_5 \propto N_c$ so that $\gamma, \rho \propto N_c^0$.

2.2 The Static Soliton Solution

Our model admits topologically non-trivial static solutions of the classical equations of motions (EOM). These are identified with the baryons and therefore the topological charge

$$B = \frac{1}{32\pi^2}\int d^3x \int_{z_{\rm UV}}^{z_{\rm IR}} dz\, \epsilon_{\hat\mu\hat\nu\hat\rho\hat\sigma}\,{\rm Tr}\left[L^{\hat\mu\hat\nu}L^{\hat\rho\hat\sigma} - R^{\hat\mu\hat\nu}R^{\hat\rho\hat\sigma}\right]\,, \tag{8}$$

is identified with the baryon number. The indices $\hat\mu$, $\hat\nu$, ... label, throughout the paper, the 4 spatial coordinates, but they are raised with Euclidean metric.

Regular static solutions with $B = 1$ have been found in [9]. The non-vanishing components of the R fields can be written in terms of 2D fields as

$$\begin{cases} \overline{R}^a_j(\mathbf{x}, z) = A_1(r, z)\widehat{x}_a\widehat{x}_j + \dfrac{1}{r}\varepsilon_{ajk}\widehat{x}_k - \dfrac{\phi_{(x)}}{r}\varepsilon^{(x,y)}\Delta^{(y),aj}\,, \\ \overline{R}^a_5(\mathbf{x}, z) = A_2(r, z)\widehat{x}^a\,, \\ \alpha\widehat{\overline{R}}_0(\mathbf{x}, z) = \dfrac{s(r, z)}{r}\,, \end{cases} \tag{9}$$

where $r^2 = \sum_i x^i x^i$, $\widehat{x}^i = x^i/r$, $\varepsilon^{(x,y)}$ is the antisymmetric tensor with $\varepsilon^{(1,2)} = 1$ and the "doublet" tensors $\Delta^{(1,2)}$ are

$$\Delta^{(x),ab} = \begin{bmatrix} \epsilon^{abc}\widehat{x}^c \\ \widehat{x}^a\widehat{x}^b - \delta^{ab} \end{bmatrix}\,. \tag{10}$$

Due to parity invariance $\{L \leftrightarrow R, \mathbf{x} \leftrightarrow -\mathbf{x}\}$ we restrict, in both the static and non–static case which we will consider in the next section, to configurations for which $L_i(\mathbf{x}, z, t) = -R_i(-\mathbf{x}, z, t)$, $L_{5,0}(\mathbf{x}, z, t) = R_{5,0}(-\mathbf{x}, z, t)$ and analogously for $\hat L$, $\hat R$. Eq. (9) therefore defines the static solution completely.

It is important to remark that the static solution in eq. (9) is "cylindrically" symmetric, meaning that it is invariant under the simultaneous action of 3D space rotations $x_a\sigma^a \to r^\dagger x_a\sigma^a r$, with $r \in SU(2)$, and vector $SU(2)$ global transformations $L, R \to r\,(L, R)\,r^\dagger$. An equivalent way to state this is that a 3D rotation with r acts on the solution (9) exactly as an $SU(2)$ vector one in the opposite direction (*i.e.* with $r^\dagger$) would do.

2.3 Zero-Mode Fluctuations

Let us now consider time-dependent infinitesimal deformations of the static solutions. Among these, the zero-mode (*i.e.* zero frequency) fluctuations are particularly important as they will describe single-baryon states. Zero-modes can be defined as directions in the field space in which uniform and slow motion is permitted by the classical dynamics and they are associated with the global symmetries of the problem, which are in our case $U(2)_V$ and 3-space rotations plus 3-space translations. The latter would describe baryons moving with uniform velocity and therefore can be ignored in the computation of static properties like the form factors. Of course, the global $U(1)_V$ acts trivially on all our fields and the global $SU(2)_V$ has the same effect as 3-space rotations on the static solution (9) because of the cylindrical symmetry. The space of static solutions which are of interest for us is therefore parametrized by 3 real coordinates –denoted as collective coordinates– which define an $SU(2)$ matrix U.

To construct zero-modes fluctuations we consider collective coordinates with general time dependence, *i.e.* we perform a global $SU(2)_V$ transformation on the static solution

$$R_{\hat{\mu}}(\mathbf{x}, z; U) \;=\; U\, \overline{R}_{\hat{\mu}}(\mathbf{x}, z)\, U^\dagger \;, \qquad \widehat{R}_0(\mathbf{x}, z; U) \;=\; \overline{\widehat{R}}_0(\mathbf{x}, z)\,, \tag{11}$$

but we allow $U = U(t)$ to depend on time. It is only for constant U that eq. (11) is a solution of the time-dependent EOM. For infinitesimal but non-zero rotational velocity

$$K = k_a \sigma^a/2 = -iU^\dagger dU/dt\,,$$

eq. (11) becomes an infinitesimal deformation of the static solution. Along the zero-mode direction uniform and slow motion is classically allowed, for this reason our fluctuations should fulfill the time-dependent EOM at linear order in K provided that $dK/dt = 0$.

From the action (4,5) the following EOM are derived

$$\begin{cases} D_{\hat{\nu}}\left(a(z) R^{\hat{\nu}}_0\right) + \dfrac{\gamma\alpha L}{4}\epsilon^{\hat{\nu}\hat{\omega}\hat{\rho}\hat{\sigma}} R_{\hat{\nu}\hat{\omega}}\widehat{R}_{\hat{\rho}\hat{\sigma}} = 0 \\ \alpha\partial_{\hat{\nu}}\left(a(z)\widehat{R}^{\hat{\nu}}{}_0\right) + \dfrac{\gamma L}{4}\epsilon^{\hat{\nu}\hat{\omega}\hat{\rho}\hat{\sigma}}\left[\mathrm{Tr}\left(R_{\hat{\nu}\hat{\omega}}R_{\hat{\rho}\hat{\sigma}}\right) + \dfrac{1}{2}\widehat{R}_{\hat{\nu}\hat{\omega}}\widehat{R}_{\hat{\rho}\hat{\sigma}}\right] = 0 \\ D_{\hat{\nu}}\left(a(z) R^{\hat{\nu}\hat{\mu}}\right) - a(z) D_0 {R_0}^{\hat{\mu}} - \dfrac{\gamma\alpha L}{2}\epsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}\left[R_{\hat{\nu}0}\widehat{R}_{\hat{\rho}\hat{\sigma}} + R_{\hat{\nu}\hat{\rho}}\widehat{R}_{\hat{\sigma}0}\right] = 0 \\ \alpha\partial_{\hat{\nu}}\left(a(z)\widehat{R}^{\hat{\nu}\hat{\mu}}\right) - \alpha a(z)\partial_0 {\widehat{R}_0}{}^{\hat{\mu}} - \gamma L\epsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}\left[\mathrm{Tr}\left(R_{\hat{\nu}0}R_{\hat{\rho}\hat{\sigma}}\right) + \dfrac{1}{2}\widehat{R}_{\hat{\nu}0}\widehat{R}_{\hat{\rho}\hat{\sigma}}\right] = 0 \end{cases} . \tag{12}$$

We only need to specify the EOM for one chirality since we are considering, as explained in the previous section, a parity invariant ansatz. We would like to find solutions of eq. (12)

for which $R_{\hat{\mu}}$ and $\widehat{R}_0$ are of the form (11); it is easy to see that the time-dependence of U in eq. (11) acts as a source for the components R_0 and $\widehat{R}_{\hat{\mu}}$, which therefore cannot be put to zero as in the static case. Notice that the same happens in the case of the 4D skyrmion [7], in which the temporal and spatial components of the ρ and ω mesons are turned on in the rotating skyrmion solution. Also, it can be shown that eq. (12) can be solved, to linear order in K and for $dK/dt = 0$, by the ansatz in Eq. (11) if the fields R_0 and $\widehat{R}_{\hat{\mu}}$ are chosen to be linear in K. Even though K must be constant for the EOM to be solved, it should be clear that this does not imply any constraint on the allowed form of the collective coordinate matrix $U(t)$ in eq. (11), which can have an arbitrary dependence on time. What we actually want to do here is to find an appropriate functional dependence of the fields on $U(t)$ such that the time-dependent EOM would be solved if and only if the rotational velocity $K = -iU^\dagger dU/dt$ was constant.

In order to solve the time-dependent equations (12) we will consider a 2D ansatz obtained by a generalization of the cylindrical symmetry of the static case. The ansatz for $R_{\hat{\mu}}$ and $\widehat{R}_0$ is specified by eq. (11) in which the static fields are given by eq. (9). Due to the cylindrical symmetry of the static solution the fields in eq. (11) are invariant under 3D space rotations $x_a\sigma^a \to r^\dagger x_a\sigma^a r$ combined with vector $SU(2)$ global transformations $L, R \to r\,(L,R)\,r^\dagger$ if U also transforms as $U \to r^\dagger U r$. We are therefore led to consider a generalized cylindrical symmetry under which k_a also rotates as the space coordinates do. Compatibly with this symmetry and with the fact that R_0 and $\widehat{R}_{\hat{\mu}}$ must be linear in K we write the ansatz as

$$R_0(\mathbf{x}, z; U) \,=\, U\,\overline{R}_0(\mathbf{x}, z; K)\,U^\dagger \,+\, i\,U\partial_0 U^\dagger\,, \qquad \widehat{R}_{\hat{\mu}}(\mathbf{x}, z; U) \,=\, \widehat{\overline{R}}_{\hat{\mu}}(\mathbf{x}, z; K)\,, \tag{13}$$

where

$$\begin{cases} \overline{R}^a_0(\mathbf{x}, z; K) = \chi_{(x)}(r,z) k_b \Delta^{(x),ab} + v(r,z)(k\cdot\widehat{x})\widehat{x}^a \\ \alpha\widehat{\overline{R}}_i(\mathbf{x}, z; K) = \dfrac{\rho(r,z)}{r}\left(k^i - (k\cdot\widehat{x})\widehat{x}^i\right) + B_1(r,z)(k\cdot\widehat{x})\widehat{x}^i + Q(r,z)\epsilon^{ibc}k_b\widehat{x}_c \\ \alpha\widehat{\overline{R}}_5(\mathbf{x}, z; K) = B_2(r,z)(k\cdot\widehat{x}) \end{cases} . \tag{14}$$

It should be noted that the term $i\,U\partial_0 U^\dagger \,=\, UKU^\dagger$ in eq. (13) is purely conventional as it could have been reabsorbed in the definition of $\overline{R}_0$. Nevertheless this choice makes manifest that our ansatz (11,13) can be obtained from the "barred" fields in eq. (9,14), which only depend on U through K, by performing a time-dependent $SU(2)$ vector gauge transformation with parameter $U(t)$. This is useful because the action, including the CS term, is invariant under this transformation. We can therefore obtain the 2D EOM for our ansatz fields by plugging the barred fields, instead of the original ones, into the 5D EOM. It is important to stress that the ansatz with barred fields is not truly gauge equivalent to

the original one because the transformation $U(t)$ does not reduce to the identity at the UV boundary, implying that the UV condition (3) is not invariant. Our true ansatz is therefore provided by eq. (11,13) and the use of the barred field as we will do in the following is just a useful trick.

At this point it is straightforward to find the zero-mode solution. The EOM for the 2D fields can be obtained by plugging the ansatz in eq. (12), while the conditions at the IR and UV boundaries are derived from eq. (2) and (3), respectively. The boundary conditions at $r = 0$ are obtained by imposing the regularity of the ansatz, while those for $r \to \infty$ come from requiring the energy of the solution to be finite and the topological charge B in eq. (8) to be equal to 1. More details are presented in appendix A, where the 2D EOM and the boundary conditions are derived. Once the 2D equations have been found, however, it is not yet trivial to solve them numerically, the procedure we followed is described in appendix B. The reader not interested in detail can however simply accept that a solution of eq. (12) exists and is given by our ansatz for some particular functional form of the 2D fields which we are able to determine numerically. In the rest of the paper the 2D fields will always denote this numerical solution of the 2D equations.

2.4 The Lagrangian of Collective Coordinates

The collective coordinate matrix $U(t)$ will be associated with static baryons. The classical dynamics of the collective coordinates is obtained by plugging eq. (11,13) in the 5D action. One finds $S[U] = \int dtL$ where

$$L = -M + \frac{\lambda}{2} k_a k^a \,. \tag{15}$$

The mass M and the moment of inertia λ are given respectively by

$$\begin{aligned} M = 8\pi M_5 \int_0^\infty dr \int_{z_{\rm UV}}^{z_{\rm IR}} dz \, \Big\{ a(z) \Big[|D_{\bar\mu}\phi|^2 + \frac{1}{4} r^2 A_{\bar\mu\bar\nu}^2 + \frac{1}{2r^2}\left(1-|\phi|^2\right)^2 - \frac{1}{2}\left(\partial_{\bar\mu} s\right)^2 \Big] \\ -\frac{\gamma L}{2}\frac{s}{r}\epsilon^{\bar\mu\bar\nu}\Big[\partial_{\bar\mu}(-i\phi^* D_{\bar\nu}\phi + h.c.) + A_{\bar\mu\bar\nu}\Big]\Big\} \,, \end{aligned} \tag{16}$$

and

$$\begin{aligned} \lambda \, = \, 16\pi M_5 \frac{1}{3}\int_0^\infty dr \int_{z_{\rm UV}}^{z_{\rm IR}} dz \, \Big\{ a(z) \Big[-\left(D_{\bar\mu}\rho\right)^2 - r^2\left(\partial_{\bar\mu} Q\right)^2 - 2Q^2 - \frac{r^2}{4} B_{\bar\mu\bar\nu} B_{\bar\mu\bar\nu} \\ + r^2\left(D_{\bar\mu}\chi\right)^2 + \frac{r^2}{2}\left(\partial_{\bar\mu} v\right)^2 + \left(\chi_{(x)}\chi_{(x)} + v^2\right)\left(1+\phi_{(x)}\phi_{(x)}\right) - 4v\phi_{(x)}\chi_{(x)} \Big] \\ + \gamma L \Big[-2\epsilon^{\bar\mu\bar\nu} D_{\bar\mu}\rho \, \chi_{(x)} \left(D_{\bar\nu}\phi\right)_{(x)} + 2\epsilon^{\bar\mu\bar\nu}\partial_{\bar\mu}\left(r\,Q\right)\,\chi_{(x)}\epsilon^{(xy)}\left(D_{\bar\nu}\phi\right)_{(y)} \\ -v\left(\frac{1}{2}\epsilon^{\bar\mu\bar\nu} B_{\bar\mu\bar\nu}\left(\phi_{(x)}\phi_{(x)} - 1\right) + r\,Q\epsilon^{\bar\mu\bar\nu} A_{\bar\mu\bar\nu}\right) + \frac{2r\,Q}{\alpha^2}\epsilon^{\bar\mu\bar\nu} D_{\bar\mu}\rho\,\partial_{\bar\nu}\left(\frac{s}{r}\right)\Big]\Big\} \,. \end{aligned} \tag{17}$$

The notations used in the equations above are defined in appendix A; the covariant derivative symbols, in particular, are associated with two Abelian residual gauge symmetries which our 2D ansatz has. Here we simply want to show that M and λ could be easily computed, at a given point of the parameter space, once the numerical solution for the 2D fields is known, by performing a numerical 2D integral.

Let us give some more detail on this theory. For now we proceed at the classical level and we will discuss the quantization in the next section. Our Lagrangian can be rewritten as

$$L \;=\; -M + \lambda \mathrm{Tr}\left[\dot{U}^\dagger \dot{U}\right] \;=\; -M \;+\; 2\lambda \sum_i \dot{u}_i^2 \,, \tag{18}$$

where we have parametrized the collective coordinates matrix U as $U = u_0 \mathbb{1} + i\, u_i \sigma^i$, with $\sum_i {u_i}^2 = 1$. The Lagrangian (18) is the one of the classical spherical rigid rotor. The variables $\{u_0, u_i\}$ are restricted to the unitary sphere S^3, which is conveniently parametrized by the coordinates $q^\alpha \equiv \{x, \phi_1, \phi_2\}$ –which run in the $x \in [-1,1]$, $\phi_1 \in [0, 2\pi)$ and $\phi_2 \in [0, 2\pi)$ domains– as

$$u_1 + i\, u_2 \;\equiv\; z_1 \;=\; \sqrt{\frac{1-x}{2}} e^{i\,\phi_1} \,, \qquad u_0 + i\, u_3 \;\equiv\; z_2 \;=\; \sqrt{\frac{1+x}{2}} e^{i\,\phi_2} \,, \tag{19}$$

where we also introduced the two complex coordinates $z_{1,2}$. We can now rewrite the Lagrangian as

$$L \;=\; -M \;+\; 2\lambda\, g_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta \,, \tag{20}$$

where g is the metric of S^3 which reads in our coordinates

$$ds^2 \;=\; g_{\alpha\beta} dq^\alpha dq^\beta \;=\; \frac{1}{4}\frac{1}{1-x^2}\, dx^2 + \frac{1-x}{2}\, d{\phi_1}^2 + \frac{1+x}{2}\, d{\phi_2}^2 \,. \tag{21}$$

The conjugate momenta are $p_\alpha = \partial L / \partial \dot{q}^\alpha = 4\lambda g_{\alpha\beta} \dot{q}^\beta$ and therefore the classical Hamiltonian is

$$H_c \;=\; M \;+\; \frac{1}{8\lambda} p_\alpha g^{\alpha\beta}(q) p_\beta \,. \tag{22}$$

It should be noted that the points U and $-U$ in what we denoted as the space of collective coordinates actually describe the same field configuration (see eq. (11,13)). The $SU(2) = S_3$ manifold we are considering is actually the universal covering of the collective coordinate space which is given by S_3/Z_2. This will be relevant when we will discuss the quantization.

3 Static properties of Nucleons

3.1 Skyrmions Quantization

We should now quantize the classical theory described above, by replacing as usual the classical momenta p_α with the differential operator $-i\partial/\partial q^\alpha$ acting on the wave functions $f(q)$. Given that the metric depends on q, however, there is an ambiguity in how to extract the quantum Hamiltonian H_q from the classical one in eq. (22). This ambiguity is resolved by requiring the quantum theory to have the same symmetries that the classical one had. At the classical level, we have an $SO(4) \simeq SU(2) \times SU(2)$ symmetry under $U \to U \cdot r^\dagger$ and $U \to g \cdot U$ with $r, g \in SU(2)$. These correspond, respectively, to rotations in space and to isospin (*i.e.* global vector) transformations, as one can see from the ansatz in eq.s (11,13). This is because K is invariant under left multiplication by g, and that the ansatz is left unchanged by performing a rotation $x_a \sigma^a \to r^\dagger x_a \sigma^a r$ and simultaneously sending $U \to U \cdot r$. The spin and isospin operators must be given, in the quantum theory, by the generators of these transformations on the space of wave functions $f(q)$ which are defined by

$$[S^a, U] = U\sigma^a/(2)\,, \qquad [I^a, U] = -\sigma^a/(2)U\,. \tag{23}$$

After a straightforward calculation one finds

$$\begin{cases} S^3 &= -\frac{i}{2}\left(\partial_{\phi_1} + \partial_{\phi_2}\right) \\ S^+ &= \frac{1}{\sqrt{2}} e^{i(\phi_1+\phi_2)} \left[i\sqrt{1-x^2}\partial_x + \frac{1}{2}\sqrt{\frac{1+x}{1-x}}\partial_{\phi_1} - \frac{1}{2}\sqrt{\frac{1-x}{1+x}}\partial_{\phi_2} \right] \\ S^- &= \frac{1}{\sqrt{2}} e^{-i(\phi_1+\phi_2)} \left[i\sqrt{1-x^2}\partial_x - \frac{1}{2}\sqrt{\frac{1+x}{1-x}}\partial_{\phi_1} + \frac{1}{2}\sqrt{\frac{1-x}{1+x}}\partial_{\phi_2} \right] \end{cases}$$
$$\begin{cases} I^3 &= -\frac{i}{2}\left(\partial_{\phi_1} - \partial_{\phi_2}\right) \\ I^+ &= -\frac{1}{\sqrt{2}} e^{i(\phi_1-\phi_2)} \left[i\sqrt{1-x^2}\partial_x + \frac{1}{2}\sqrt{\frac{1+x}{1-x}}\partial_{\phi_1} + \frac{1}{2}\sqrt{\frac{1-x}{1+x}}\partial_{\phi_2} \right] \\ I^- &= -\frac{1}{\sqrt{2}} e^{-i(\phi_1-\phi_2)} \left[i\sqrt{1-x^2}\partial_x - \frac{1}{2}\sqrt{\frac{1+x}{1-x}}\partial_{\phi_1} - \frac{1}{2}\sqrt{\frac{1-x}{1+x}}\partial_{\phi_2} \right] \end{cases} \tag{24}$$

where the raising/lowering combinations are $S^\pm = (S^1 \pm iS^2)/\sqrt{2}$.

The operators in eq. (24) should obey the Hermiticity conditions $(S^3)^\dagger = S^3$, $(S^+)^\dagger = S^-$, and analogously for the isospin. In order for the Hermiticity conditions to hold we choose the scalar product to be

$$\langle A|B\rangle \equiv \int d^3q\, \sqrt{g} f_A{}^\dagger(q) f_B(q)\,, \tag{25}$$

where $\sqrt{g} = 1/4$ in our parametrization of S_3. The reason why this choice of the scalar product gives the correct Hermiticity conditions is that S^a and I^a (where $a = 1, 2, 3$) can be written as $X^\alpha \partial_\alpha$ with X^α Killing vectors of the appropriate S_3 isometries. The Killing equation $\nabla_\alpha X_\beta + \nabla_\beta X_\alpha = 0$ ensures the generators to be Hermitian with respect to the scalar product (25).

Knowing that the scalar product must be given by eq. (25) greatly helps in guessing what the quantum Hamiltonian, which has to be Hermitian, should be. We can multiply and divide by $\sqrt{g}$ the kinetic term of H_c and move one $\sqrt{g}$ factor to the left of p_α. Then we apply the quantization rules and find [5]

$$H_q = M - \frac{1}{8\lambda}\frac{1}{\sqrt{g}}\partial_\alpha\left(\sqrt{g}g^{\alpha\beta}\partial_\beta\right) = M - \frac{1}{8\lambda}\nabla_\alpha\nabla^\alpha\,, \tag{26}$$

which is clearly Hermitian. We can immediately show that H_q commutes with spin and isospin, so that the quantum theory is really symmetric as required: a straightforward calculation gives indeed

$$H_q = M + \frac{1}{2\lambda}S^2 = M + \frac{1}{2\lambda}I^2\,. \tag{27}$$

It would not be difficult to solve the eigenvalue problem for the Hamiltonian (26), but in order to find the nucleon wave functions it is enough to note that the versor of n-dimensional Euclidean space provides the n representation of the $SO(n)$ isometry group. In our case, $n = 4 = (2, 2)$, which is exactly the spin/isospin representation in which nucleons live. It is immediately seen that z_1, as defined in eq. (19), has $S^3 = I^3 = 1/2$. Acting with the lowering operators we easily find the wave functions

$$\begin{aligned} |p\uparrow\rangle &= \frac{1}{\pi}z_1\,, & |n\uparrow\rangle &= \frac{i}{\pi}z_2\,, \\ |p\downarrow\rangle &= -\frac{i}{\pi}\bar{z}_2\,, & |n\downarrow\rangle &= -\frac{1}{\pi}\bar{z}_1\,, \end{aligned} \tag{28}$$

which are of course normalized with the scalar product (25). The mass of the nucleons is therefore $E = M + 3/(8\lambda)$.

Notice that the nucleon wave functions are odd under $U \to -U$, meaning that they are double-valued on the genuine collective coordinate space S_3/Z_2. This corresponds, following [19], to quantize the skyrmion as a fermion and explains how we could get spin-1/2 states after a seemingly bosonic quantization without violating spin–statistic.

Let us now summarize some useful identities which will be used in our calculation. First of all, it is not hard to check that, after the quantization is performed the rotational velocity

[5] The last equality holds because H_q is supposed to be acting on the wave functions, which are scalar functions.

becomes

$$k^a = -i \operatorname{Tr}\left[U^\dagger \dot{U} \sigma^a\right] = \frac{1}{\lambda} S^a \,, \tag{29}$$

and analogously

$$i \operatorname{Tr}\left[\dot{U} U^\dagger \sigma^a\right] = \frac{1}{\lambda} I^a \,. \tag{30}$$

Other identities which we will use in our calculations are

$$\begin{aligned} \langle \operatorname{Tr}\left[U\, \sigma^b U^\dagger \sigma^a\right] &= -\frac{8}{3} S^b I^a \rangle \,, \\ \langle \operatorname{Tr}\left[U\, \sigma^b \widehat{x}_b (k \cdot \widehat{x}) U^\dagger \sigma^a\right] &= -\frac{2}{3\lambda} I^a \rangle \,, \end{aligned} \tag{31}$$

where the VEV symbols $\langle ... \rangle$ mean that those are not operatorial identities, but they only hold when the operators act on the subspace of nucleon states. Notice that the second equation in (31) is implied by the first one if one also uses the commutation relation (23), eq. (30) and the fact that, on nucleon states, $\langle \{S^a, S^i\} = \delta^{ai}/2 \rangle$.

3.2 The Nucleon Form Factors

The nucleon form factors parametrize the matrix element of the currents on two nucleon states. For the isoscalar and isovector currents we have

$$\begin{aligned} \langle N_f(p')|J^\mu_S(0)|N_i(p)\rangle &= \bar{u}_f(p') \left[F_1^S(q^2)\gamma^\mu + \frac{iF_2^S(q^2)}{2M_N}\sigma^{\mu\nu}q_\nu\right] u_i(p), \\ \langle N_f(p')|J^{\mu a}_V(0)|N_i(p)\rangle &= \bar{u}_f(p') \left[F_1^V(q^2)\gamma^\mu + \frac{iF_2^V(q^2)}{2M_N}\sigma^{\mu\nu}q_\nu\right] (2I^a)\, u_i(p), \end{aligned} \tag{32}$$

where the currents are defined as $J_V^a = J_R^a + J_L^a$ and $J_S = 1/3\left(\widehat{J}_R + \widehat{J}_L\right)$ in terms of the chiral ones. In the equation above $q \equiv p' - p$ is the 4-momentum transfer, N_i and N_f are the initial and final nucleon states and $u_i(p)$, $\bar{u}_f(p')$ their wave functions, $I^a = \sigma^a/2$ is the isospin generators and $\sigma^{\mu\nu} \equiv i/2[\gamma^\mu, \gamma^\nu]$. For the axial current $J_A^a = J_R^a - J_L^a$ we have

$$\langle N_f(p')|J^a_{A\mu}(0)|N_i(p)\rangle = \bar{u}_f(p') G_A(q^2) \left[\gamma_\mu - \frac{2M_N}{q^2} q^\mu\right] \gamma^5 I^a u_f(p) \,. \tag{33}$$

Exact axial and isospin symmetries, which hold in our model, have been assumed in the definitions above.

In our non-relativistic model the current correlators will be computed in the Breit frame in which the initial nucleon has 3-momentum $-\vec{q}/2$ and the final $+\vec{q}/2$ (i.e. $p^\mu = (E, -\vec{q}/2)$ and $p'^\mu = (E, \vec{q}/2)$, and $q^2 = -\vec{q}^{\,2}$, with $E = \sqrt{M_N^2 + \vec{q}^{\,2}/4}$). Notice that the textbook definitions in eq.s (32,33) involve nucleon states which are normalized with $\sqrt{2E}$; in order

to match with our non-relativistic normalization we have to divide all correlators by $2M_N$. The vector currents become

$$
\begin{aligned}
\langle N_f(\vec{q}/2)|J_S^0(0)|N_i(-\vec{q}/2)\rangle &= G_E^S(\vec{q}^{\,2})\chi_f^\dagger\chi_i\,,\\
\langle N_f(\vec{q}/2)|J_S^i(0)|N_i(-\vec{q}/2)\rangle &= i\,\frac{G_M^S(\vec{q}^{\,2})}{2M_N}\chi_f^\dagger 2(\vec{S}\times\vec{q})^i\chi_i\,,\\
\langle N_f(\vec{q}/2)|J_V^{0a}(0)|N_i(-\vec{q}/2)\rangle &= G_E^V(\vec{q}^{\,2})\chi_f^\dagger\,(2I^a)\,\chi_i\,,\\
\langle N_f(\vec{q}/2)|J_V^{ia}(0)|N_i(-\vec{q}/2)\rangle &= i\,\frac{G_M^V(\vec{q}^{\,2})}{2M_N}\chi_f^\dagger 2(\vec{S}\times\vec{q})^i\,(2I^a)\,\chi_i\,,
\end{aligned}
\tag{34}
$$

where we defined

$$
G_E^{S,V}(-q^2) = F_1^{S,V}(q^2) + \frac{q^2}{4M_N^2}F_2^{S,V}(q^2)\,,\qquad G_M^{S,V}(-q^2) = F_1^{S,V}(q^2) + F_2^{S,V}(q^2)\,,
\tag{35}
$$

and used the definition $(\vec{S}\times\vec{q})^i \equiv \varepsilon^{ijk}S^jq^k$. The nucleon spin/isospin vectors of state $\chi_{i,f}$ are normalized to $\chi^\dagger\chi = 1$. For the axial current we find

$$
\begin{aligned}
\langle N_f(\vec{q}/2)|J_A^{i,a}(0)|N_i(-\vec{q}\,/2)\rangle &= \chi_f^\dagger\frac{E}{M_N}G_A(\vec{q}^{\,2})2S_T^iI^a\chi_i\,,\\
\langle N_f(\vec{q}\,/2)|J_A^{0,a}(0)|N_i(-\vec{q}\,/2)\rangle &= 0
\end{aligned}
\tag{36}
$$

where $\vec{S}_T \equiv \vec{S} - \hat{\vec{q}}\;\vec{S}\cdot\hat{\vec{q}}$ is the transverse component of the spin operator.

It is straightforward to compute the matrix elements of the currents in position space on static nucleon states. Plugging the ansatz (9,11,14,13) in the definition of the currents (6) and performing the quantization one obtains quantum mechanical operators acting on the nucleons. The matrix elements are easily computed using the results of sect. 3.1. We finally obtain the form factors by taking the Fourier transform and comparing with eq.s (34,36). We have [6]

$$
\begin{aligned}
G_E^S &= -\frac{N_c}{6\pi\gamma L}\int dr\,r\,j_0(qr)\,(a(z)\partial_z s)_{UV}\\
G_E^V &= \frac{4\pi M_5}{3\lambda}\int dr\,r^2\,j_0(qr)\,\Big[a(z)\,\Big(\partial_z v - 2\,(D_z\chi)_{(2)}\Big)\Big]_{UV}\\
G_M^S &= \frac{8\pi M_N M_5\alpha}{3\lambda}\int dr\,r^3\,\frac{j_1(qr)}{qr}\,(a(z)\partial_z Q)_{UV}\\
G_M^V &= \frac{M_N\,N_c}{3\pi L\gamma\alpha}\int dr\,r^2\,\frac{j_1(qr)}{qr}\,\Big(a(z)\,(D_z\phi)_{(2)}\Big)_{UV}\\
G_A &= \frac{M_N}{E}\frac{N_c}{3\pi\alpha\gamma L}\int dr\,r\,\Big[a(z)\frac{j_1(qr)}{qr}\,\Big((D_z\phi)_{(1)} - r\,A_{zr}\Big) - a(z)\,(D_z\phi)_{(1)}\,j_0(qr)\Big]_{UV}
\end{aligned}
\tag{37}
$$

where j_n are spherical Bessel functions which arise because of the Fourier transform.

[6] It is quite intuitive that the form factors can be computed in this way. Given that solitons are infinitely heavy at small coupling, in the Breit frame they are almost static during the process of scattering with the current. To check this, however, we should perform the quantization of the collective coordinates associated with the center-of-mass motion, as it was done in [20] for the original 4D Skyrme model.

4 Results

In this section we will present our results. After discussing some qualitative features, such as the large-N_c scaling of the form factors and the divergences of the isovector radii due to exact chiral symmetry, we extrapolate to the physically relevant case of $N_c = 3$ and perform a quantitative comparison with the experimental data. Consistently with our working hypothesis that the 5D model really describes large-N_c QCD we find a 30% relative discrepancy.

Large-N_c Scaling

Let us take all the three parameters α, γ and L of our 5D model to scale like N_c^0 for large-N_c. Eq. (7) therefore implies that the coupling M_5 grows like N_c and the semiclassical expansion in 5D coincides with the $1/N_c$ expansion on the 4D side. Notice that these scaling of the parameters are uniquely dictated by what we know to be the large-N_c scaling of meson couplings and masses. In the baryon sector, the solitonic solution is independent of N_c given that M_5 factorizes out of the action and does not appear in the EOM. The classical mass M and the moment of inertia λ therefore scale like N_c and the scaling of the form factors can be easily read from eq. (37).

In large-N_c QCD the scaling of several baryon observables is known [21]. The mass grows with N_c as in our model, but this is expected to be a common feature of any soliton model. The matrix element of currents on normalized nucleon states should be of the form $N_c^p F(q^2)$ with $p = 1$ even though cancellations, *i.e.* $p < 1$, are not excluded. All the radii should therefore scale like N_c^0 and this is what we find in our model. We also find the "naive" –*i.e.* with $p = 1$– overall scaling for the electric scalar (G_E^S), magnetic vector (G_M^V) and axial (G_A) form factor; notice that, due to the definition in eq. (34), the magnetic form factors scale with one more power of N_c than what the corresponding current matrix element does. We however find two cancellations: due to the $1/\lambda$ factor the electric vector G_E^V and the magnetic scalar G_M^S scale like N_c^0 and N_c^1, respectively. This corresponds to a "$p = 0$" scaling of the associated currents.

The reason for the cancellation in G_E^V is very simple to understand. Remembering that the temporal component of the current at zero momentum gives the conserved charge and looking at the definitions (34) one immediately obtains two consistency conditions: $G_E^S(0) = N_c/6$, because in the nucleons there are N_c quarks which have $U(1)_V$ charge $1/6$ each in our conventions, and $G_E^V(0) = 1/2$, because nucleons are in the $1/2$ representation of isospin. It

is not difficult to see that these consistency conditions are respected by our model as they are implied by the EOM, and they are fulfilled to great accuracy (0.1%) by the numerical solution. The above discussion implies, in particular, that while the electric scalar form factor G_E^S has the naive N_c scaling, the electric vector G_E^V does not.

We are not able to prove that the cancellation in G_M^S actually takes place in large-N_c QCD, but we can check that it occurs in the naive quark model, or better in its generalization for arbitrary odd $N_c = 2\,k{+}1$ [22]. In this non-relativistic model the Nucleon wave function is made of $2k{+}1$ quark states, $2k$ of which are collected into k bilinear spin/isospin singlets while the last one has free indices which give to the Nucleon its spin/isospin quantum numbers. Of course, the wave function is symmetrized in flavor and spin given that the color indices are contracted with the antisymmetric tensor and the spatial wave function is assumed to be symmetric. The current operator is the sum of the currents for the $2k+1$ quarks, each of which will assume by symmetry the same form as in eq. (34). If $S_{1,2}$ and $I_{1,2}$ represent the spin and isospin operators on the quarks $q_{1,2}$ the operators $S_1 + S_2$ and $I_1 + I_2$ will vanish on the singlet combination of the two quarks, but $S_1 I_1 + S_2 I_2$ will not. The k singlets will therefore only contribute to G_E^S, G_M^V and G_A, which will have the naive scaling, while for the others we find cancellations.

A detailed calculation can be found in [23] where, among other things, the proton and neutron magnetic moments and the axial coupling are computed in the naive quark model. The magnetic moments are related to the form factor at zero momentum as $\mu_V/\mu_N = G_M^V(0)$ and $\mu_S/\mu_N = G_M^S(0)$ where $\mu_N = 1/(2M_N)$ is the nuclear magneton and $2\,\mu_V = \mu_p - \mu_n$, $2\,\mu_S = \mu_p + \mu_n$. In accordance with the previous discussion, the results in the naive quark model are $2\mu_S = \mu_u + \mu_d$ and $2\mu_V = 2k/3(\mu_u - \mu_d)$, where $\mu_{u,d}$ are the quark magnetic moments, while for the axial coupling one finds $g_A = G_A(0) = 2k/3 + 1 = N_c/3 + 2/3$ which scales like N_c as expected. Notice that the $1/N_c$ corrections to the axial coupling are quite big in the naive quark model: for $N_c = 3$ the leading term contributes as 1 while the "true" results is 5/3, which is 67% bigger. We have of course no reason to believe that such big corrections should persist in the true large-N_c expansion of QCD, this trivial remark simply suggests that "large" $1/N_c$ corrections to the form factors are not excluded.

Divergences in the Chiral Limit

It is well known that in QCD the isovector electric $\langle r^2_{E,V}\rangle$ and magnetic $\langle r^2_{M,V}\rangle$ radii which are proportional, respectively, to the q^2 derivative of G_E^V and G_M^V at zero momentum, diverge in the chiral limit [24]. If the nucleons are effectively described by an isospin doublet of

point-like spinors added to the $\chi_{\mathbf{PT}}$ Lagrangian this effect comes from pion loops which are IR divergent in the massless pion limit $m_\pi \to 0$ [25]. It is not obvious, however, that the divergences should survive in the large-N_c limit, *i.e.* that they should already appear in the leading term of the perturbative $1/N_c$ expansion. This is so because describing baryons as weakly coupled particles, which is a reasonable approximation in real-world $N_c = 3$ QCD, is not possible at large-N_c given that their coupling with pions grows like $N_c^{3/2}$. It was noticed in [26] that, in a model which only contains the nucleon doublet, the pion–nucleon scattering amplitude grows like N_c violating unitarity and also contradicting the usual large-N_c counting rules. The theory is therefore inconsistent and the full infinite tower of large-N_c baryons must be added. Moreover, it is very easy to see that the results of [24, 25] are not compatible with large N_c: the one-loop corrections to the radii, which are of course finite for finite m_π, have the wrong scaling and grow like N_c. Given that we cannot apply the results of [24, 25], we cannot conclude that the radii must diverge in our model at the leading order in the semiclassical expansion of the soliton, but there is of course no problem if they do. We must however check that all the other radii are finite, and this is what we will do in the following. What we will find is the same as in the 4D Skyrme model [6]: all radii and form factors are finite but the electric and magnetic isovector ones.

In our model, as in the Skyrme model, divergences in the integrals of eq. (37) which define the form factors are due, as in QCD, to the massless pions. If all the fields were massive, indeed, any solution to the EOM would fall down exponentially at large r while in the present case power-like behaviors can appear. These power-like terms in the large-r expansion of the solution can be derived analytically by performing a Taylor expansion of the fields around infinity ($1/r = 0$), substituting into the EOM and solving order by order in $1/r$. The exponentially suppressed part of the solution will never contribute to the expansion. In the gauge in which (the form factors are, of course, gauge invariant) the topological twist is at the origin $r = 0$ and the solution is trivial for $r \to \infty$ the first few terms are [7]

$$\begin{cases} A_1 = \dfrac{2z(z-1)}{r^3}\beta \\ A_2 = \dfrac{\beta}{r^2} + \dfrac{4z^3 - 6z^2 + 1}{2r^4}\beta \\ \phi_1 = \dfrac{z(1-z)}{r^2}\beta + \dfrac{z(z^3 - 2z^2 + 1)}{2r^4}\beta \\ \phi_2 = -1 + \dfrac{z^2(3-2z)}{2r^4}\beta^2 \\ s = \dfrac{z^2(z^6 - 4z^4 + 8)}{4r^8}\gamma\beta^3 \end{cases} \qquad \begin{cases} \chi_1 = \dfrac{z(z-1)}{r^2}\beta \\ \chi_2 = 1 + \dfrac{z^2(2z-3)}{2r^4}\beta^2 \\ v = -1 + \dfrac{z^2(z^2-3)^2}{12r^6}\beta^2 \\ q = -\dfrac{z^2(z^6 - 4z^4 + 8)}{4r^8}\gamma\beta^3 \end{cases} \tag{38}$$

where β is an unknown parameter which depends on the entire solution and can only be

[7] In the equations which follow we put $L = 1$ for simplicity.

determined numerically. We checked that the large-r behavior of our numerical solution is very well approximated by eq. (38). Substituting these expressions into the definitions of the form factors (37) one gets

$$\begin{cases} G_E^S \propto \beta^3 \int dr \frac{1}{r^7} j_0(qr) + \ldots \\ G_E^V \propto \beta^2 \int dr \frac{1}{r^2} j_0(qr) + \ldots \\ G_M^S \propto \beta^3 \int dr \frac{1}{r^5} \frac{j_1(qr)}{qr} + \ldots \\ G_M^V \propto \beta^2 \int dr \frac{1}{r^2} \frac{j_1(qr)}{qr} + \ldots \end{cases} . \tag{39}$$

All the form factors are finite for any q, including $q = 0$. The electric and magnetic radii, however, are defined as

$$\langle r^2_{E,M} \rangle = -\frac{6}{G_{E,M}(\vec{q}^2 = 0)} \left. \frac{dG_{E,M}(\vec{q}^2)}{d\vec{q}^2} \right|_{\vec{q}^2=0} , \tag{40}$$

and taking a q^2 derivative of eq.s (39) makes one more power of r^2 appear in the integral. It is easy to see that the scalar radii are finite, while the vector ones are divergent as anticipated. We will now discuss the axial coupling and the axial radius and show that both are finite.

The expression in eq. (37) for the axial form factor G_A presents some subtleties for vanishing q^2. Given the asymptotic expansion of the solution in eq. (38) the axial coupling integral behaves for large r like

$$G_A \propto \int dr \left[\left(\frac{3}{r}\beta - \frac{1}{r^5}\beta^3 \right) \frac{j_1(qr)}{qr} + \left(-\frac{1}{r}\beta + \frac{5}{7r^5}\beta^3 \right) j_0(qr) + \ldots \right] , \tag{41}$$

where the leading $1/r$ terms (the ones which are linear in β) can be obtained from eq. (38) while for the others one needs higher order terms which are not reported in eq. (38). The integral in eq. (41) is finite integral for any $q \neq 0$. For $q \to 0$, however, the integral is not uniformly convergent and one cannot exchange the limit with the integration. The leading $1/r$ term in eq. (41) is indeed given by $I(q) = \beta \int_0^\infty dr \; (1/r) \left(3j_1(qr)/(qr) - j_0(qr)\right)$ which is independent of q and equal to $\beta/3$. Given that the argument of the integral vanishes for $q \to 0$ exchanging the limit and integral operations would give the wrong result $I(0) = 0$. To restore uniform convergence and obtain an analytic formula for g_A one can subtract the $I(q)$ term from the expression in eq. (37) for G_A and replace it with $\beta/3$. Rewriting the axial form factor in this way is also useful to establish that the axial radius, which seems divergent if looking at eq. (41), is on the contrary finite. The $I(q)$ term, indeed, does not contribute to the q^2 derivative and the ones which are left in eq. (41) give a finite contribution.

We have found, in summary, that all the form factors and radii are finite but the isovector ones. Given that the divergences are related with the large-distance behaviors of the fields, and that our model reduces to the Skyrme model in the IR, this result is not surprising. A different result has been found, however, in Ref. [13], where the nucleon form factors have been computed in the Sakai–Sugimoto model. In that case all the radii are finite. The Sakai–Sugimoto baryons correspond, as explained in the Introduction, to the small-size limit of the 5D skyrmions we are considering, and we should recover the results of [13] if we perform a small-γ expansion which correspond to the $1/\lambda$ expansion considered in [13]. As γ decreases our soliton becomes more and more localized around $(r = 0, z = z_{\rm IR})$ and at any large but fixed value of r the deviations from the pure-gauge configuration become smaller and smaller. The small-γ expansion of the asymptotic solution (38) therefore coincides with the small-β expansion. By looking at eq. (39) we see that the power-like terms in the isoscalar and isovector form factor densities appear at high orders in β and this explains why these densities were found to be exponentially damped in [13]. For the axial form factor, as eq. (41) shows, power-like terms are present at the linear order in β. The same term has been found in [13] but it does not lead to any divergence as explained in the previous paragraph.

A possible physical explanation of the finiteness of the radii in the Sakai–Sugimoto model is that the 5D soliton effectively reduces to a 5D particle in the limit of small size, a possibility discussed in [28]. For a 5D particle no divergences appear in the radii at the leading order in the semiclassical expansion (*i.e.* at tree level) and the divergences should arise, in analogy with the case of a 4D particle, at loop level. Following the analogy, however, one could expect the divergent loop corrections (or better the enhanced loop corrections for small but finite pion mass) to have, as it happens for the 4D particle, the wrong large-N_c scaling. By the same reasoning one could expect unitarity violation in the pion-nucleon scattering amplitude at tree-level.

Pion Form Factor and Goldberger–Treiman relation

It is of some interest to define and compute the pion-nucleon form factor which parametrizes the matrix element on Nucleon states of the pion field. In the Breit frame (for normalized nucleon states) it is

$$\langle N_f(\vec{q}/2)|\pi^a(0)|N_i(-\vec{q}/2)\rangle = -\frac{i}{2M_N\vec{q}^2}G_{NN\pi}(\vec{q}^2)\chi_f^\dagger(2S^i)q_i(2I^a)\chi_i\,, \tag{42}$$

where $\pi^a(x)$ is the normalized and "canonical" pion field operator. The field is canonical in the sense that its quadratic effective Lagrangian only contains the canonical kinetic term $\mathcal{L}_2 = 1/2(\partial\pi_a)^2$, or equivalently that its propagator is the canonical one, without a non-trivial

form factor. With this definition, $G_{NN\pi}$ is the vertex form factor of the meson-exchange model for nucleon-nucleon interactions [29] and corresponds to an interaction [8]

$$\mathcal{L}_{NN\pi} = i\,(G_{NN\pi}(\Box)\pi_a)\overline{N}\gamma^\mu\gamma_5(2I^a)N\,. \tag{43}$$

On-shell, the form factor reduces to the pion-nucleon coupling constant, $G_{NN\pi}(0) = g_{NN\pi}$, whose experimental value is $g_{NN\pi} = 13.5 \pm 0.1$.

The pion field which matches the requirements above is given by the zero-mode of the KK decomposition. In the unitary gauge $\partial_z(a(z)A_5) = 0$, where $A_M \equiv (L_M - R_M)/2$, and for AdS$_5$ space $a(z) = L/z$ one has

$$A_5^{(un)}(x,z) = \frac{1}{F_\pi L}\frac{1}{a(z)}\pi^a(x)\sigma_a\,, \tag{44}$$

with $F_\pi^2 = 2M_5/\int dz/a(z) = 4M_5/L$. [9] Gauge-transforming back to the gauge in which our numerical solution is provided and using the ansatz in eq.s (9,11) we find the pion field

$$\pi^a \;=\; -\frac{F_\pi}{2}\int_{z_{\rm UV}}^{z_{\rm IR}} dz A_2(r,z)\widehat{x}^b\mathrm{Tr}\left[U\sigma_b U^\dagger\sigma^a\right]\,. \tag{45}$$

Taking the matrix element of the above expression and comparing with eq. (42) one obtains

$$G_{NN\pi}(q^2) = -\frac{8\pi}{3}M_N F_\pi q\int_0^\infty dr j_1(qr)\int dz\, r^2\, A_2(r,z)\,. \tag{46}$$

Using eq. (38) it is easy to understand that the $q \to 0$ limit of $G_{NN\pi}$ is completely determined by the large-r behavior of the field A_2, and in particular by the leading β/r^2 term. Due to the q factor, indeed, only the divergent part of the integral contributes. We then find

$$g_{NN\pi} = -\frac{2N_C}{3\pi}\frac{M_N}{F_\pi\gamma\alpha}\beta\,. \tag{47}$$

We used the formula above to check numerically that the Goldberger–Treiman relation $F_\pi g_{\pi NN} = M_N g_A$ holds in our model, we find that it is verified to 0.01% on our numerical solution. We can also demostrate the Goldberger–Treiman relation by using eq. (33,34) of [6] which show that also g_A is determined by the asymptotic behavior of the axial current. We indeed obtain

$$g_A = -\frac{2N_C}{3\pi\alpha\gamma}\beta\,. \tag{48}$$

[8]Nucleon scattering, in our model, is a soliton scattering process and we have no reason to believe that it can be described by meson-exchange, *i.e.* that contact terms are suppressed. Therefore, we will not attempt any comparison of our form factor with the one used in meson-exchange models.

[9]We take the opportunity here to remark that the formula for F_π reported in [9], though written for general warp factor $a(z)$, is only correct in the case of AdS$_5$ space in which $a(z) = L/z$.

	Experiment	AdS_5	Deviation
M_N	940 MeV	1130 MeV	20%
μ_S	0.44	0.34	30%
μ_V	2.35	1.79	31%
g_A	1.25	0.70	79%
$\sqrt{\langle r^2_{E,S}\rangle}$	0.79 fm	0.88 fm	11%
$\sqrt{\langle r^2_{E,V}\rangle}$	0.93 fm	∞	
$\sqrt{\langle r^2_{M,S}\rangle}$	0.82 fm	0.92 fm	12%
$\sqrt{\langle r^2_{M,V}\rangle}$	0.87 fm	∞	
$\sqrt{\langle r^2_{A}\rangle}$	0.68 fm	0.76 fm	12%
μ_p/μ_n	-1.461	-1.459	0.1%

Table 1: Prediction of the nucleon observables with the microscopic parameters fixed by a fit on the mesonic observables. The deviation from the empirical data is computed using the expression $|th - exp|/\min(|th|, |exp|)$, where th and exp denote, respectively, the prediction of our model and the experimental result.

Comparison with Experiments

Let us now compare our results with real-world QCD, we therefore fix the number of colors $N_c = 3$ and choose our microscopic parameters to be $1/L \simeq 343$ MeV, $M_5 L \simeq 0.0165$ and $\alpha \simeq 0.94$ ($\gamma \simeq 1.23$). These values are obtained by minimizing the root mean square error (RMSE) in the mesonic sector. The detailed list of the observables we used can be found in [9] and the minimum RMSE for mesons is found to be 11%.

The numerical results of our analysis and the deviation with respect to the experimental data are reported in table 1. We find a fair agreement with the experiments, a 36% total RMSE which is compatible with the expected size of $1/N_c$ corrections. We discussed in the previous section that the isovector radii are divergent because of the chiral limit, it would be interesting to add the pion mass to the model and compute these observables. Table 1 also shows the proton-neutron magnetic moment ratio, which is in perfect agreement with the experimental value. Notice that for this observable, due to the different scalings of μ_S and μ_V with N_c, our computation includes two orders of the $1/N_c$ expansion: the leading order value which is -1 and the next-to-leading $1/N_c$ correction which accounts for the extra -0.46. The axial charge is the one which shows the larger (almost 100%) deviation, and indeed removing this observable the RMSE decreases to 21%. We cannot exclude that, in a theory in which the naive expansion parameter is 1/3, enhanced 80% relative corrections to few observables might appear at the next-to-leading order. This failure in g_A, therefore,

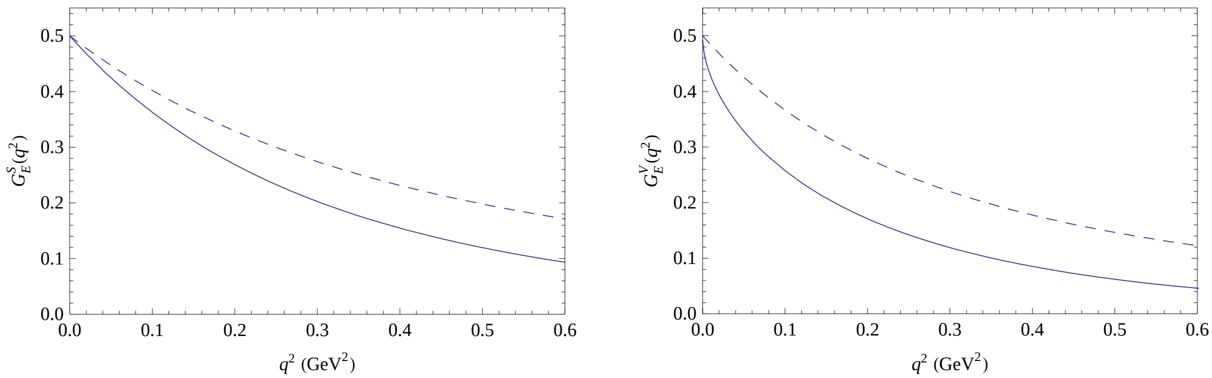

Figure 1: Scalar (left) and vector (right) electric form factors. We compare the results with the empirical dipole fit (dashed line) [7].

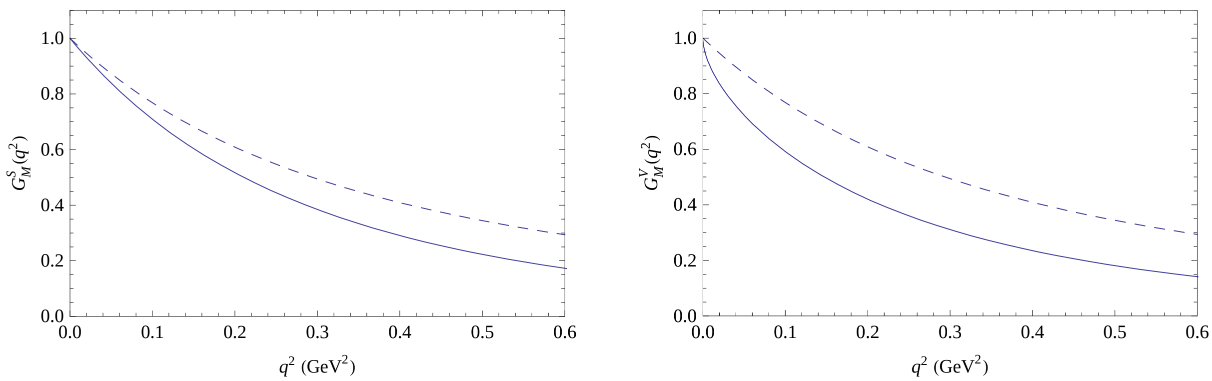

Figure 2: Normalized scalar (left) and vector (right) magnetic form factors. We compare the results with the empirical dipole fit (dashed line) [7].

does not invalidate the general picture.

It is interesting to notice that a much better prediction for g_A is obtained if one uses, instead of the standard procedure [6] considered in this paper, a different approach to the quantization of collective coordinates of the skyrmion, which has been proposed in Ref. [17]. The results of Ref. [17] can be directly applied to our case since, for what concerns the collective coordinate quantization, the 5D nature of our soliton is immaterial. We therefore find that the prediction for μ_S and for the radii are unaffected while both μ_V and g_A are rescaled by 5/3. We still obtain a good prediction for $\mu_V = 2.98$ (which is 27% away from the experimental value) and a much better prediction for $g_A = 1.17$. Being the quantization of [17] equivalent to the standard one at large–N_c, we have no reasons to prefer, a priori, one or the other. We have no reason either, however, to believe that the $1/N_c$ corrections one includes in this alternative approach really capture the leading $1/N_c$ corrections or at least part of them. If this was the case we should, of course, use the non–standard quantization and the discrepancy in the prediction of g_A would disappear.

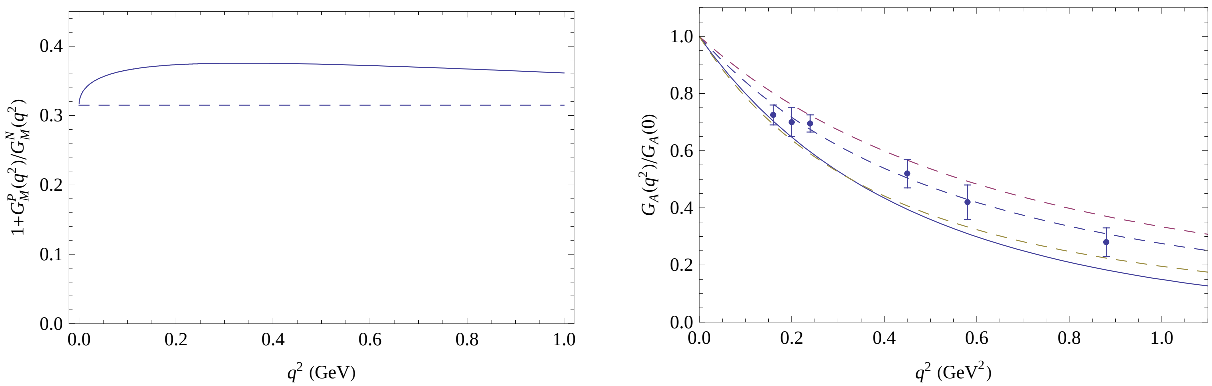

Figure 3: Left: deviation of the ratio of proton and neutron magnetic form factors from the large N_c value (solid line), compared with the dipole fit of the experimental data (dashed line). Right: normalized axial form factor (solid line) compared with the empirical dipole fit (dashed lines) [7] and with the experimental data taken from [30, 31].

If we stick, on the contrary, to the standard quantization procedure a small value of g_A ($g_A = 0.65$ [6]) is also obtained in the original Skyrme model, but the situation improves if the effects of the ρ and ω mesons are taken into account. The "complete" model described in Ref. [7] seems the one which should better mimic our 5D scenario, and $g_A = 0.99$ in that case. The explicit chiral symmetry breaking, which is turned on in [7], could explain the difference because the axial coupling is strongly sensitive to the large-r behavior of the solution (see the discussion following eq. (41)) which is in turn heavily affected by the presence of the pion mass. Correction to g_A from chiral symmetry breaking could therefore be enhanced. Notice that, however, this expectation fails in the original Skyrme model, where the addition of the pion mass does not affect g_A significantly [27].

In figs. 1, 2 and 3 we compare the normalized nucleon form factors at $q^2 \neq 0$ with the dipole fit of the experimental data. The shape of the scalar and axial form factors is of the dipole type, the discrepancy is mainly due to the error in the radii. The shape of vector form factors is of course not of the dipole type for small q^2, but this is due to the divergence of the derivative at $q^2 = 0$. Including the pion mass will for sure improve the situation given that it will render finite the slope at zero momentum; it would be interesting to see if the dipole shape of these form factors is recovered in the presence of the pion mass. We also plot in the left panel of fig. 3 the deviation of ratio of the proton and neutron magnetic form factors from the large N_c value which is given, due to the the different large-N_c scaling of the isoscalar and isovector components, by $G_M^P(q)/G_M^N(q) = -1$. Not only we find that this quantity is quite well predicted, with an error $\lesssim 15\%$, but we also see that its shape, in agreement with observations, is nearly constant away from $q^2 = 0$. Also in this case correction from the pion mass are expected to go in the right direction.

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A The Equations of Motion

In this technical appendix we report the EOM for the 2D fields which appear in our ansatz in eqs. (9) and (14) and we explain the notation used throughout the paper.

The Residual Gauge Invariance

Before discussing the detailed form of the EOM, it is useful to observe that our ansatz has not fixed the 5D gauge freedom completely, its form is indeed preserved by chiral $SU(2)_{L,R}$ gauge transformations of the form $g_R = U(t) \cdot g \cdot U^\dagger(t)$ and $g_L = U(t) \cdot g^\dagger \cdot U^\dagger(t)$ with

$$g = \exp[i\alpha(r,z)x^a\sigma_a/(2r)]\,. \tag{49}$$

The operators Δ defined in eq. (10) have simple transformation rules under the residual symmetry. Indeed

$$\begin{aligned} g\Delta^{(1),ab}\sigma_a/2g^\dagger &= \cos\alpha\Delta^{(1),ab}\sigma_a/2 + \sin\alpha\Delta^{(2),ab}\sigma_a/2\,,\\ g\Delta^{(2),ab}\sigma_a/2g^\dagger &= \cos\alpha\Delta^{(2),ab}\sigma_a/2 - \sin\alpha\Delta^{(1),ab}\sigma_a/2\,, \end{aligned}$$

so that the 2D fields $\phi_{(x)}$ and $\chi_{(x)}$ defined respectively in eq. (9) and (14) transform as charged complex scalars under this residual $U(1)$. It is not hard to see that the fields $A_{\bar\mu} = \{A_1, A_2\}$ transform as gauge field, so that under a residual transformation one has

$$\begin{cases} A_{\bar\mu} \to A_{\bar\mu} + \partial_{\bar\mu}\alpha(r,z)\,,\\ \phi \equiv \phi_1 + i\phi_2 \to e^{i\alpha(r,z)}\phi\,,\\ \chi \equiv \chi_1 + i\chi_2 \to e^{i\alpha(r,z)}\chi\,, \end{cases} \tag{50}$$

while all the other fields are invariant.

There is also a second residual $U(1)$ associated with chiral $U(1)_{L,R}$ 5D transformations of the form $\widehat{g}_R = \widehat{g}$ and $\widehat{g}_L = \widehat{g}^\dagger$ with

$$\widehat{g} = \exp\left[i\beta(r,z)\frac{(k\cdot\widehat{x})}{\alpha}\right] . \tag{51}$$

Under this second residual $U(1)$ only $B_{\bar{\mu}} = \{B_1, B_2\}$ and ρ transform non trivially. We have

$$\begin{cases} B_{\bar{\mu}} \to B_{\bar{\mu}} + \partial_{\bar{\mu}}\beta\,, \\ \rho \to \rho + \beta\,, \end{cases} \tag{52}$$

and therefore $B_{\bar{\mu}}$ is a gauge field and ρ a Goldstone.

In order to make manifest the residual gauge invariance in the action and the EOM we introduced gauge covariant derivatives for the ϕ, χ and ρ fields

$$\begin{cases} (D_{\bar{\mu}}\phi)_{(x)} = \partial_{\bar{\mu}}\phi_{(x)} + \epsilon^{(xy)}A_{\bar{\mu}}\phi_{(y)} \\ (D_{\bar{\mu}}\chi)_{(x)} = \partial_{\bar{\mu}}\chi_{(x)} + \epsilon^{(xy)}A_{\bar{\mu}}\chi_{(y)} \\ D_{\bar{\mu}}\rho = \partial_{\bar{\mu}}\rho - B_{\bar{\mu}} \end{cases} . \tag{53}$$

The Equations of Motion

The easiest way to derive the EOM for the 2D fields is to start from the Lagrangian and substitute the ansatz. Using the 5D action in eqs. (4) and (5) and rewriting the 5D fields in terms of the 2D ones (eqs. (9), (11), (13) and (14)), after a straightforward computation one finds the expressions for the mass M and moment of inertia λ given in eqs. (16) and (17). Notice that in order to obtain the order K^2 terms of the action one has to perform a symmetric integration in d^3x, which can simply be implemented by the replacement $\widehat{x}^i\widehat{x}^j \to 1/3\delta^{ij}$. We report here two contraction identities of the “doublet” operators Δ (eq. (10)) which can be useful for the computation of the 2D action

$$\Delta^{(x),ab}\Delta^{(y),ac} = -\delta^{(xy)}\Delta^{(2),bc} + \epsilon^{(xy)}\Delta^{(1),bc}\,, \tag{54}$$

$$\Delta^{(x),ab}\epsilon^{bid}\widehat{x}_d = \epsilon^{(xy)}\Delta^{(y),ai}\,. \tag{55}$$

The EOM for the 2D fields can be simply obtained, at this point, by imposing the variation of the 2D action to vanish. We have also checked the consistency of our ansatz by showing that the same EOM are obtained by substituting directly into the 5D equations (12). The EOM for the fields which are already turned on in the static case are

$$\begin{cases} D^{\bar{\mu}}\left(a(z)D_{\bar{\mu}}\phi\right) + \dfrac{a(z)}{r^2}\phi(1-|\phi|^2) + i\gamma L\epsilon^{\bar{\mu}\bar{\nu}}\partial_{\bar{\mu}}\left(\dfrac{s}{r}\right)D_{\bar{\nu}}\phi = 0 \\ \partial^{\bar{\mu}}\left(r^2 a(z)A_{\bar{\mu}\bar{\nu}}\right) - a(z)\left(i\phi^\dagger D_{\bar{\nu}}\phi + h.c.\right) + \gamma L\epsilon^{\bar{\mu}\bar{\nu}}\partial_{\bar{\mu}}\left(\dfrac{s}{r}\right)(|\phi|^2-1) = 0 \\ \partial_{\bar{\mu}}\left(a(z)\partial^{\bar{\mu}}s\right) - \dfrac{\gamma L}{2r}\epsilon^{\bar{\mu}\bar{\nu}}\left[\partial_{\bar{\mu}}(-i\phi^\dagger D_{\bar{\nu}}\phi + h.c.) + A_{\bar{\mu}\bar{\nu}}\right] = 0 \end{cases} , \tag{56}$$

while the equations for the “new” fields which are turned on for the rotating skyrmion are

$$\begin{cases} \partial^{\bar\mu}(r^2 a(z)\partial_{\bar\mu} v) - 2a(z)\left[v(1+|\phi|^2) - \chi\phi^\dagger - \phi\chi^\dagger\right] + \gamma L \epsilon^{\bar\mu\bar\nu}\left[\frac{1}{2}(|\phi|^2-1)B_{\bar\mu\bar\nu} + rQA_{\bar\mu\bar\nu}\right] = 0 \\ D^{\bar\mu}(r^2 a(z) D_{\bar\mu}\chi) + a(z)\left[2v\phi - (1+|\phi|^2)\chi\right] - \gamma L\epsilon^{\bar\mu\bar\nu}(D_{\bar\mu}\phi)\left[i\partial_{\bar\nu}(rQ) + D_{\bar\nu}\rho\right] = 0 \\ \frac{1}{r}\partial^{\bar\mu}(r^2 a(z)\partial_{\bar\mu} Q) - \frac{2}{r}a(z)Q \\ \qquad -\frac{\gamma L}{2}\epsilon^{\bar\mu\bar\nu}\left[(iD_{\bar\mu}\phi(D_{\bar\nu}\chi)^\dagger + h.c.) + \frac{1}{2}A_{\bar\mu\bar\nu}(2v - \chi\phi^\dagger - \phi\chi^\dagger) - \frac{2}{\alpha^2}D_{\bar\mu}\rho\,\partial_{\bar\nu}\left(\frac{s}{r}\right)\right] = 0 \\ \partial_{\bar\mu}(a(z)D_{\bar\mu}\rho) - \frac{\gamma L}{2}\epsilon^{\bar\mu\bar\nu}\left[\left(D_{\bar\mu}\phi(D_{\bar\nu}\chi)^\dagger + h.c.\right) + \frac{i}{2}A_{\bar\mu\bar\nu}(\phi\chi^\dagger - \chi\phi^\dagger) + \frac{2}{\alpha^2}\partial_{\bar\mu}(rQ)\partial_{\bar\nu}\left(\frac{s}{r}\right)\right] = 0 \\ \partial^{\bar\nu}\left(r^2 a(z) B_{\bar\nu\bar\mu}\right) + 2a(z)D_{\bar\mu}\rho \\ \qquad +\gamma L\epsilon^{\bar\mu\bar\nu}\left\{\left[(\chi - v\phi)(D_{\bar\nu}\phi)^\dagger + h.c.\right] + (1-|\phi|^2)\partial_{\bar\nu}v - \frac{2r}{\alpha^2}Q\,\partial_{\bar\nu}\left(\frac{s}{r}\right)\right\} = 0 \end{cases} \tag{57}$$

In order to solve numerically the EOM, they must be rewritten as a system of elliptic partial differential equations. This can be achieved by choosing a 2D Lorentz gauge condition for the residual $U(1)$ gauge fields

$$\partial^{\bar\mu} A_{\bar\mu} = 0\,, \qquad \partial^{\bar\mu} B_{\bar\mu} = 0\,. \tag{58}$$

In this way the equations for $A_{\bar\nu}$ become $J^{\bar\nu} = \partial_{\bar\mu}\left(r^2 a A^{\bar\mu\bar\nu}\right) = r^2 a \partial_{\bar\mu}\partial^{\bar\mu} A^{\bar\nu} + \partial_{\bar\mu}(r^2 a) A^{\bar\mu\bar\nu}$ which is an elliptic equation and a similar result is obtained for $B_{\bar\mu}$. As discussed in [8], to impose the gauge condition, one can solve the “gauge-fixed” EOM counting the gauge field components as independent fields. In this way, if one imposes the gauge conditions at the boundaries, then the gauge is maintained also in the bulk.

The Boundary Conditions

The IR and UV boundary conditions on the 2D fields follow from eq. (2) and eq. (3) and from the gauge choice in eq. (58). They are given explicitly by

$$z = z_{\rm IR}\ :\quad \begin{cases} \phi_1 = 0 \\ \partial_2\phi_2 = 0 \\ A_1 = 0 \\ \partial_2 A_2 = 0 \\ \partial_2 s = 0 \end{cases} \qquad \begin{cases} \chi_1 = 0 \\ \partial_2\chi_2 = 0 \\ \partial_2 v = 0 \\ \partial_2 Q = 0 \end{cases} \qquad \begin{cases} \rho = 0 \\ B_1 = 0 \\ \partial_2 B_2 = 0 \end{cases}\,, \tag{59}$$

and

$$z = z_{\rm UV}\ :\quad \begin{cases} \phi_1 = 0 \\ \phi_2 = -1 \\ A_1 = 0 \\ \partial_2 A_2 = 0 \\ s = 0 \end{cases} \qquad \begin{cases} \chi_1 = 0 \\ \chi_2 = -1 \\ v = -1 \\ Q = 0 \end{cases} \qquad \begin{cases} \rho = 0 \\ B_1 = 0 \\ \partial_2 B_2 = 0 \end{cases}\,. \tag{60}$$

The boundary conditions at $r = \infty$ have to ensure that the energy of the solution is finite, this means that the fields should approach a pure-gauge configuration. At the same time one has to require that the solution is non-trivial and its topological charge (eq. (8)) is non zero. To obtain a soliton solution with $B = 1$ one can impose the conditions

$$r = \infty \quad : \quad \begin{cases} \phi = -ie^{i\pi z/L} \\ \partial_1 A_1 = 0 \\ A_2 = \frac{\pi}{L} \\ s = 0 \end{cases} \qquad \begin{cases} \chi = ie^{i\pi z/L} \\ v = -1 \\ Q = 0 \end{cases} \qquad \begin{cases} \rho = 0 \\ \partial_1 B_1 = 0 \\ B_2 = 0 \end{cases} . \tag{61}$$

The $r = 0$ boundary of our domain requires an ad hoc treatment, given that the EOM become singular there. Of course this boundary is not a true boundary of our 5D space, but it represents some internal points. Thus we must require the 2D solution to give rise to regular 5D vector fields at $r = 0$ and we must also require the gauge choice to be fulfilled. These conditions are

$$r = 0 \quad : \quad \begin{cases} \phi_1/r \to A_1 \\ (1+\phi_2)/r \to 0 \\ A_2 = 0 \\ \partial_1 A_1 = 0 \\ s = 0 \end{cases} \qquad \begin{cases} \chi_1 = 0 \\ \chi_2 = -v \\ \partial_1 \chi_2 = 0 \\ Q = 0 \end{cases} \qquad \begin{cases} \rho/r \to B_1 \\ \partial_1 B_1 = 0 \\ B_2 = 0 \end{cases} . \tag{62}$$

B Numerical Techniques

To obtain the numerical solution of the EOM we used the COMSOL 3.4 package [32], which permits to solve a generic system of differential elliptic equations by the finite elements method. A nice feature of this software is that it allows us to extend the domain up to boundaries where the EOM are singular (*i.e.* the $r = 0$ line), because it does not use the bulk equations on the boundaries, but, instead, it imposes the boundary conditions.

In order to improve the convergence of the program and the numerical accuracy, one is forced to perform a coordinate and a field redefinition. The former is needed to include the $r = \infty$ boundary in the domain in which the numerical solution is computed. The advantage of this procedure is the fact that in this way one can correctly enforce the right behaviour of the fields at infinity by imposing the $r = \infty$ boundary conditions. A convenient coordinate change is given by

$$x = c \arctan\left(\frac{r}{c}\right) , \tag{63}$$

where x is the new coordinate used in the program and c is an arbitrary constant. The domain in the x direction is now reduced to the interval $[0, c\pi/2]$. The parameter c has

been introduced to improve the numerical convergence of the solution. A good choice for c is $c \sim 10$, which allows to have a reasonable domain for x and, at the same time, does not compress the solution towards $x = 0$.

A field redefinition is needed to impose the regularity conditions at $r = 0$ (eq. (62)). For this purpose we use the rescaled fields

$$\begin{cases} \phi_1 = x\psi_1 \\ \phi_2 = -1 + x\psi_2 \\ \rho = x\tau \end{cases} . \tag{64}$$

With these redefinitions, in the new coordinates, the $r = 0$ boundary conditions read as

$$r = 0 \;:\quad \begin{cases} \psi_1 - A_1 = 0 \\ \psi_2 = 0 \\ A_2 = 0 \\ \partial_x A_1 = 0 \end{cases} \qquad \begin{cases} \chi_1 = 0 \\ \partial_x \chi_2 = 0 \\ v = -\chi_2 \\ Q = 0 \end{cases} \qquad \begin{cases} \tau - B_1 = 0 \\ \partial_x B_1 = 0 \\ B_2 = 0 \end{cases} . \tag{65}$$

In order to ensure the convergence of the program another modification is needed. As already discussed, to obtain a soliton solution with non-vanishing topological charge we have to impose non-trivial boundary conditions for the 2D fields at $r = \infty$ (eq. (61)). It turns out that, if such conditions are imposed, the program is not able to reach a regular solution. This is so because the $r = \infty$ boundary is singular and imposing non-trivial (though gauge-equivalent to the trivial ones) boundary conditions at a singular point spoils the regularity of the numerical solution; the same would happen if the topological twist was located at $r = 0$. To fix this problem we have to perform a gauge transformation which reduces the $r = \infty$ conditions to trivial ones and preserves the ones at $r = 0$ at the cost of introducing a "twist" on the UV boundary. For this, we use a transformation of the residual $U(1)$ chiral gauge symmetry associated to $SU(2)_{L,R}$ (eq. (49)) with

$$\alpha(r, z) = (1 - z/L) f(r)\,, \tag{66}$$

where $f(r)$ can be an arbitrary function which respects the conditions

$$\begin{cases} f(0) = 0 \\ f(\infty) \to \pi \end{cases} \qquad \text{and} \qquad \begin{cases} f''(0) = 0 \\ f''(\infty) \to 0 \end{cases} . \tag{67}$$

For $c \sim 10$ it turns out that a good choice for $f(r)$ is $f(r) = 2 \arctan r$. The gauge-fixing condition for $A_{\bar{\mu}}$ is now modified as

$$\partial_r A_1 + \partial_z A_2 - (1 - z/L) f''(r) = 0\,, \tag{68}$$

the UV boundary conditions are given by

$$z = z_{\rm UV} \;\; : \quad \begin{cases} x\psi_1 = \sin f(r) \\ (-1 + x\psi_2) = -\cos f(r) \\ A_1 = f'(r) \\ \partial_z A_2 = 0 \\ s = 0 \end{cases} \qquad \begin{cases} \chi_1 = -\sin f(r) \\ \chi_2 = \cos f(r) \\ v = -1 \\ Q = 0 \end{cases} \qquad \begin{cases} \tau = 0 \\ B_1 = 0 \\ \partial_z B_2 = 0 \end{cases} , \tag{69}$$

and the $r = \infty$ constraints are now trivial

$$r = \infty \;\; : \quad \begin{cases} \psi_1 = 0 \\ (-1 + x\psi_2) = 1 \\ \partial_x A_1 = 0 \\ A_2 = 0 \\ s = 0 \end{cases} \qquad \begin{cases} \chi = -i \\ v = -1 \\ Q = 0 \end{cases} \qquad \begin{cases} \tau = 0 \\ \partial_x B_1 = 0 \\ B_2 = 0 \end{cases} , \tag{70}$$

whereas the $r = 0$ and the IR boundary conditions are left unchanged. Notice that in the new gauge the EOM for $A_{\bar{\mu}}$ are modified in accord to eq. (68), however they are still in the form of elliptic equations.

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